

XI. *A Letter from the Rev. Patrick Murdoch, F. R. S. concerning the mean Motion of the Moon's Apogee, to the Rev. Dr. Robert Smith, Master of Trinity College Cambridge.*

Reverend Sir,

Read Jan. 31.  
1750. LAST summer, when I was to pay my respects to you at Trinity College, I gave you some account of the warm dispute, then lately arisen between Mr. de Buffon and Mr. Clairaut, two eminent academicians at Paris; the latter pretending, that the Newtonian law of attraction is inconsistent with the motion of the moon's apogee; and that its quantity ought not to be expressed by  $\frac{1}{x^2}$  of the distance, but by two, or perhaps more, terms of a series, as  $\frac{1}{x^2} + \frac{a}{x^4}$ . Which new doctrine Mr. Clairaut had got inserted in the memoirs of the academy, and Mr. de Buffon had followed him close with another memoir, confuting it.

When I first heard of this controversy, it was impossible to judge of the validity of Mr. Clairaut's reasons, because he kept his *calculus* a profound secret. But an absurd consequence of his new law of attraction occurr'd to me, as soon as Mr. de Buffon mention'd the thing, that, "if we should put the attraction, express'd by his two terms, of an assumed quantity *G*, and resolve the equation, there would necessarily



" necessarily arise two different values of the distance  
 "  $x$ , for the same attractive force."

Suspecting therefore, that some error must have slipped into Mr. Clairaut's reasonings (as he himself afterwards found there had) I resolved to try, whether, by an arithmetical calculation, from Sir Isaac Newton's propositions only, the motion in question might not be accounted for.

The result of this inquiry I should have taken the liberty to send you before now, but that, other things intervening, I did not think of revising and transcribing it, till lately; that Mr. Walmesley having made me a present of his ingenious treatise on the same subject, it appears, that, however Mr. Clairaut's hypothesis is given up, yet a notion still prevails, as if Sir Isaac Newton's propositions, concerning the motion of *apsids*, were mere mathematical fictions, not applicable to nature.

How far I have succeeded in shewing the contrary, is now submitted to your judgment. And I, at the same time, embrace, with pleasure, an opportunity of professing myself, with the highest respect,

Reverend Sir,

Sradishall, 6 April,

1750.

Your most obliged, and

most obedient humble servant,

Pat. Murdocke.

Of the mean motion of the moon's apogee, according to Sir Isaac Newton.

The rule given by Sir Isaac Newton, in the 9 section of his first book, is to this purpose: Tab. Fig. 3.

1. That, supposing the common law of attraction, and that a central body  $T$  attracts the body  $P$ , revolving round it in an orbit nearly circular, with a force as unity; if to this be added a constant force, whose ratio to the former is expressed by  $c$ ; then the angular velocity of the body  $P$ , in an immoveable plane, will be to its angular velocity, reckoned from the *ap**sis* of its orbit, in the subduplicate ratio of  $1+c$  to  $1+4c$ , or as  $\sqrt{\frac{1+c}{1+4c}}$  to unity. And therefore, if  $A$  represents any arc described by the revolving body in an immoveable plane,  $A \times \sqrt{\frac{1+4c}{1+c}}$  will be the corresponding arc in its orbit, reckon'd from the *ap**sis*. And their difference  $A \times \sqrt{\frac{1+4c}{1+c}} - 1$ , will be the excess of the *ap**sis*.

But if the force of the central body  $T$  is diminished by some constant force as  $c$ , then the sign of  $c$  is changed in these expressions; and the direct motion of the *ap**sis* will be  $A \times 1 - \sqrt{\frac{1-4c}{1-c}}$ .

2. And hence, if some foreign variable force, added to, or subtracted from, the central force of attraction, produces a given motion of the *ap**sis*, retrograde or direct; it is easy to find a constant force as  $c$ , which should produce the same motion.

3. Let

3. Let  $S$  represent the sun, at an immense distance,  $T$  the earth (supposed, for the present, at rest)  $P$  the moon's place in her orbit  $ADBC$ , in which  $C, D$ , are the quadratures,  $A, B$ , the syzygies: then if  $PK$ , parallel to  $AB$ , and cutting  $TC$  in  $K$ , be produced till  $KL$  is double of  $PK$ ; and  $LM$  parallel to  $PT$  meet  $AB$  produced in  $M$ ;  $LM$  and  $MT$  will represent the disturbing forces of the sun, by which the moon is urged in the directions  $PT, MT$ . See *Princip. lib. i. prop. 66.* and *lib. iii. prop. 25, 26.*

And if  $TR$  is made perpendicular to  $LM$ , the force  $MT$  shall be resolved into two forces as  $RT$  and  $MR$ ; whereof the latter,  $MR$ , taken from  $LM$ , reduces the disturbing force, in the direction  $PT$ , to their difference  $LR$ .

4. Put now  $PT (=LM)=1$ ;  $PK$ , the sine of the arc  $PC=s$ : and then  $TM (=PL=3s)$ :  $MR::1:s$ ; that is  $MR=3s^2$ , and  $LR$ , the disturbing force in the direction  $PT$ , is as  $1-3s^2$ .

When  $\ell p$ , the moon's distance from the quadrature, is an arc of  $35^\circ 15' 52''$ , in which case  $1-33^2=0$ ,  $l$  and  $r$  coincide; and the disturbing force vanishing, the line of the *apsids* becomes stationary.

But if the moon's distance from her quadrature is still greater, as at  $\pi$ , then  $\mu\varrho$  exceeds  $\mu\lambda$ ; and their difference  $\lambda\varrho$  is a force represented by  $-1-3s^2$ , acting in the direction  $T\pi$ . This force, at the syzygies, is double of  $TC$ .

5. Whence, and from § 1, it follows; that  $c$  being the sun's disturbing force, in the direction  $CT$ , at the quadrature; at any other point, as  $P$ , it will be  $\pm c \times 1-3s^2$ . And that writing for  $c$  the variable quantity  $c \times 1-3s^2$ , and  $A$  for the fluxion of the

arc  $CP$ , the fluent of  $A \times \sqrt{\frac{1+4c \times i - 3s^2}{1+c \times i - 3s^2}}$  will give the motions of the *apsis*.

6. The quantity  $c$  being  $\frac{1000}{178725}$  of the earth's mean attractive force at the moon; by computing, as above, it will be found, that while the moon moves from  $C$  to  $p$ , through an arc of  $35^\circ 15' 52''$ , the total regress of the *apsis* is to the arc  $Cp$  as .005404 ( $=n$ ) to unity: and that the sum of its direct motions, while the moon moves from  $p$  to  $A$ , is to the arc  $pA$  as .0105707 ( $=N$ ) to unity.

It will be found likewise, by the inverse operation hinted in § 2, that putting  $k = .00362552$ , and  $K = .0069611$ ;  $+k$  and  $-K$  are forces, which acting constantly, the one from  $C$  to  $p$ , the other from  $p$  to  $A$ , would produce the same motions of the *apsis*.

7. The quantities  $k$  and  $K$  might have been found, pretty near the truth, only by summing the ordinates  $i \angle R$ , or  $i - 3s^2$ , upon the arc  $A$ : in which case we should have had  $k = c \times .648869 = .00370925$ , and  $K = c \times 1.24018 = .006939$ : and the motions thence computed would not have been much different from their just quantity. This however is mentioned, not as if the method itself were sufficiently exact; but to shew, that if, hereafter, in cases, where the limits of the forces are incomparably narrower, we shall, instead of summing the *momenta*, make use of a mean force determined in a like manner, there is no sensible error to be apprehended.

8. Hitherto we have considered the body  $T$ , round which  $P$  revolves, as quiescent; and it is thus authors have always considered it: altho' the case in nature,

to which they meant to apply Sir Isaac Newton's rule, is widely different. The earth and moon revolve about their common centre of gravity : their distances from which being inversely as their masses, and the forces, by which either is attracted by the other, as also the forces of the sun to disturb their motions, being in the same *ratio*; it follows, that the earth in her motion round the common centre of gravity will suffer disturbances every way similar to those of the moon. And the whole motion of the *apsis* of the moon's orbit, resulting from the two disturbing forces, will be near the double of what either of them could produce separately, round a fix'd centre \*.

9. To

\* " *Quatenus terra et luna circum commune gravitatis centrum revolvuntur, perturbabitur etiam motus terræ circa centrum illud a viribus consimilibus; sed summas tam virium quam motuum referre licet ad lunam.*" Princip. p. 429.

And p. 141. *Apsis lunæ est duplo velocior circiter*: but this has been strangely mistaken, as if the author having revised and printed his 9th section a third time, and above forty years after it was invented, should, after all, own, that it signified nothing to his purpose. Would this be the *nil molitur inepte*, so justly applied to Newton?

See likewise, p. 423; where having deduced the motion of the *apsids* of Jupiter's satellites from that of our moon's, he adds, "*Diminui tamen debet motus augis sic inventus in ratione 5 ad 9, vel 1 ad 2, circiter, ob causam, quam hic exponere non vacat.*"

The reason is not, that the orbits of Jupiter's moons are less eccentric than that of ours, as some have imagined; for, "*augendo vel diminuendo excentricitatem et inclinationem orbis, non mutatur motus augis sensibiliter, nisi ubi eadem sunt nimis magnæ,*" p. 180. Is it not rather, because the action of the several satellites upon their primary and upon one another, in all the possible variety of directions, reduces the case of any particular satellite to that of a single body revolving round a fix'd centre, *viz.* that of Jupiter's system?

9. To determine which, we may conceive the earth as revolving in an orbit that is already in motion from the sun's disturbing force upon the moon: the retrograde motion of the orbit, while the earth moves from  $C$  to  $p$ , being  $n \times Cp$ ; and the direct motion, for the rest of the quadrant, being  $N \times pA$ ; whence it will follow, that the disturbing force =  $k$  affects the earth's motion thro' an arc of her orbit equal to  $Cp \times \frac{1}{1+n}$ ; and the force  $-K$  acts thro' the arc  $pA \times \frac{1}{1+N}$ . And the motions of the *apsis* being in the same *ratio's*, if  $r$  is the regress of the *apsis* of the moon's orbit (determined as in § 6) and  $p$  its progress; the regress of the *apsis* of the earth's orbit will be  $r \times \frac{1}{1+n}$ , and its direct motion,  $p \times \frac{1}{1-N}$ . That is, the whole motions of the *apsis*, resulting from the sun's action upon the earth and moon together, will be ( $R=$ )  $r \times \frac{1}{2+n}$ , and ( $P=$ )  $p \times \frac{1}{2-N}$ ; and the motions to be ascribed to either arc,  $r \times \frac{1}{1+\frac{1}{2}n}$ , and  $p \times \frac{1}{1-\frac{1}{2}N}$ .

Now  $p$ , found as above, being  $2082''.9$ . and  $N=$   
 $.0105707$ ,  $P$  is  $4143''.8$ . And the same way,  $R=$   
 $1375''.7$ : whose difference  $P-R$  multiplied by 4, that is,  $4 \times 2768''=11072''=3^{\circ} 4' 32''$ , is the direct motion of the *apsis* in a revolution.

First correction for the moon's variation. Fig. 4.

10. In the foregoing calculation, it is supposed, that the moon's orbit is nearly circular, more nearly indeed than it possibly can be, even abstracting from its excentricity. For altho' the moon had been projected with a direction and force to make her de-  
 scribe

scribe a circle round the earth, as  $EOL$ , the action of the sun would have changed this orbit into an oval, as  $OADBC$ ; whose greatest diameter, passing thro' the quadratures  $CD$ , is to the least as  $70\frac{1}{4}$  to  $69\frac{1}{24}$ . The reason and determination of which we have in *Princip. lib. iii. prop. 26, 28.*

11. That this action of the sun, and the figure resulting from it, must lessen the mean motion of the apogee, is easily shewn.

For let  $P$  be the moon's place in her orbit, when the *apsis* is stationary, and  $EOL$  the circle of her mean motion, cutting the orbit very near the octant  $O$ , and  $PT$  in  $o$ : then, the accelerating forces of the earth at  $P$  and  $o$ , being inversely as the squares of  $PT$  and  $oT$ , and the sun's disturbing force at the points  $P, o$ , being in the simple direct *ratio* of the same lines;  $oT$  being given, the *ratio* of the sun's disturbing force at the point  $P$ , to the earth's accelerating force at the same point, that is, the quantity  $c$  in the theorem, will be as the cube of the distance  $PT$ : and, *a fortiori*, in every point of the orbit, from the quadrature  $C$  to  $P$ , will exceed the mean force at  $O$ , and its effect in producing a retrograde motion of the *apsis* will be greater.

For the remaining part of the quadrant, where the motion of the *apsis* is direct, the force  $c$  is indeed greater than its mean quantity from  $P$  to  $O$ ; but, thro' the whole octant  $OA$ , it is continually decreasing as the cube of the distance from  $T$ : whence, upon the whole, that force, and its effect, from  $P$  to  $A$ , fall short of their mean quantities at  $O$ . Seeing therefore the direct motion is diminished, and the retrograde increased; their difference, that is, the direct

rect motion in the quadrant  $CPA$  will be diminished.

But this mean motion will be diminished somewhat likewise from the inequable description of the areas (in *prop. 26. lib. iii.*) : on which account, the cubes of the distance  $PT$  must be every where increased or diminished in the duplicate *ratio* of the moments of time in which a given little angle is described, to the mean moment at the octant \*.

12. By computing from these principles, it will be found ;

1. That the angle  $CTP$ , which was of  $35^{\circ} 15' 52''$  in the circle, will, in the oval orbit, be diminished to  $34^{\circ} 43' 34''$ .

2. That the *ratio* of the mean of the cubes of the moon's distances in the arc  $CP$ , to the cube of the mean distance, will be express'd by  $1.023916$  ( $=g$ ) and

\* To express the distance  $PT$  by  $s$  the sine of the angle  $CTP$ , in an ellipsis not very eccentric : from any point  $P$  draw  $PK$  an ordinate to the axis  $CD$ , and meeting the circumscribed circle in  $M$ ; draw likewise  $Mf$  perpendicular to  $TP$  produced. Then putting  $TC=1$ ,  $TA=d$ ,  $\frac{1-d}{d}=t$ ; by conjoining the *ratio's* of  $TP$

to  $PK$ ,  $PK$  to  $PM$ ,  $PM$  to  $Pf$ , it will be  $TP = \frac{Tf}{1+ts^2}$  : in which

for the variable numerator  $Tf$ , we might, because of the smallness of the angle  $PTM$ , write unity : but taking it rather of its mean quantity  $m$  ( $=.999987$  in the moon's orbit) the distances, whose cubes are to be summed, will be  $\frac{m}{1+ts^2}$ .

And the *ratio* of the moments of time to the mean moment is that of  $110.23$  to  $109.73 + s^2$ , by *prop. 26. lib. iii.*

and the like *ratio*, in the arc  $PA$ , by .9852467 ( $=b$ ).

3. Multiplying therefore the forces  $k$  and  $-K$ , found in § 6, by  $g$  and by  $b$ , substituting the products for  $c$ , in the *formula*, with the arcs  $CN$ , and  $NG$ , respectively, and finishing the operation as for the circle, the regres, in a periodical month, will be  $5548''.3$ , and the progress  $16489''.8$ : whose difference is the direct mean motion sought,  $3^\circ 21'' 2\frac{1}{2}$ .

13. But nearly the same conclusion may be obtained, and with much less trouble, as follows:

In the circle  $CGD$ ; take  $CM = 35^\circ 15' 52''$ , and thro'  $P$ , the point where  $MK$  perpendicular to  $TC$ , cuts the orbit, draw  $TPN$  meeting the circle in  $N$ . Then, if  $R$  is the regres of the *apsis* in a circular orbit,  $R \times \frac{\overline{CM}}{\overline{CN}}^{\frac{1}{2}}$  will be the regres in the oval  $CPA$ .

In like manner, having inscribed in the orbit, the circle  $Amb$ , and made a similiar construction for the rest of the quadrant  $P \times \frac{\overline{Am}}{\overline{Ab}}^{\frac{1}{2}}$ , will be the direct motion in the oval,  $P$  being the direct motion in a circle.

Thus, the angle of variation  $MTN$  being (in Dr. Halley's tables)  $33' 9''$ , the subduplicate *ratio* of  $CM$  to  $CN$  will be  $1.007927$ , and that of  $Am$  to  $Ab$ , or of  $GM$  to  $GN$ , will be  $.99499$ . And therefore  $R$  (in § 9) will be augmented to  $1386''.6$ , and  $P$  diminish'd to  $4123''$ : whose difference, multiplied by 4, gives  $3^\circ 2' 25\frac{1}{2}''$ ; exceeding the former only by about  $4''$ .

14. The rule is founded in this, that if, from the centre  $T$ , a circular arc  $Ff$  be described, including in the angle  $CTN$  the sector  $FTf$ , equal to the elliptic sector  $CTP$ , the cube of  $TF$ , the *radius* of this circle, may be taken for the mean of the cubes of the moon's distances in the arc  $CP$ . And because the area  $CPT$  is to the sector  $CMT$ , as  $PK$  to  $KM$ , or as  $TA$  to  $TC$ ; and  $To$  or  $TE$  is a geometrical mean between  $TA$  and  $TC$ , it will easily appear, that  $TF^3 : To^3 :: CM_{\frac{3}{2}} : CN_{\frac{3}{2}}$ . And that  $P$ , found from the tables, being (nearly at least) the stationary point in the oval, if the force  $k$  is increased in the sesquiplicate ratio of  $CM$  to  $CN$ , and the arc  $CN$  substituted for  $A$  in the formula, we shall, by § 1, find the retrograde motion of the *apsis*.

Now, when the constant force  $+k$  is given, the regres  $R$  is as the arc  $A$ ; and when  $A$  is given, and  $k$  is but a little augmented,  $R$  is proportional to  $k$ : in general therefore, if  $k$  is but a little augmented,  $R$  is as  $k \times A$ . Write  $\mathcal{Q}$  for the regres in the oval,  $R$  standing for that in the circle, already found; and it will be  $\mathcal{Q} : R :: k \times \frac{CM}{CN}_{\frac{3}{2}} \times CN : k \times CM$ , or  $\mathcal{Q} = R \times \frac{CM}{CN}_{\frac{1}{2}}$ , according to the rule. The like reasoning for the direct motion.

Second correction for the Excentricity. Fig. 5.

15. This equation is inconsiderable; because, altho' the *ratio* of the disturbing force, when the moon is at a greater than her mean distance, is more increased than it is diminished in the opposite points of her orbit;

orbit; this increase is very near compensated by the comparative smallness of the angular velocity.

Let  $ADa$  represent the moon's elliptic orbit, whose centre is  $C$ , its axes  $Aa$ ,  $Dd$ , the mean eccentricity  $CT$ , and the circle of her mean motion  $MDmd$ , cutting  $Aa$  in  $M$  and  $m$ . Then, because it is a mean motion we seek, generated while the axis  $Aa$  passes thro' all its different aspects of the sun; we may conceive the direct motion already found, of  $3^\circ 2' 21''\frac{1}{2}$ , to be produced by a constant disturbing force  $-K$ , acting on the moon as she revolves in her circular orbit  $MDmd$ ; and we have only to enquire, how much this force, and its effects, are to be increased, the moon really moving about the same centre  $T$ , in the elliptic arc  $AD$ ; and how much diminished in the arc  $Da$ .

16. For which purpose, the constant force  $K$  is to be increased in the *ratio* of the mean of the cubes of the moon's distances, in the arc  $AD$ , to the cube of  $TD$  or  $CA$ , and diminished as the mean of the cubes of the distances in  $Da$ . Let the forces resulting be  $\dot{K} \times G$  and  $\dot{K} \times H$ ; and these being substituted in the *formula*, with the arcs  $2DM$ ,  $2Dm$ , respectively, the sum of the motions found will be the whole mean motion of the apogee, including the correction for the excentricity.

Now  $\dot{K}$  will be found to be .00557337, and the excentricity  $TC$  being .05505, and  $Q$  the quadrantal arc to radius 1; the *ratio*  $G$ , or, which is the same, the sesquiciple of the time, in which the elliptic arc  $AD$  is described, to the time in the circular arc  $DM$ , that is,  $\frac{Q+TC}{DM}^{\frac{3}{2}}$  will be 1.110942; and  $H$  ( $= \frac{Q-TC}{DM}^{\frac{3}{2}}$ )

=.9001387: whence the whole motion, found as above directed, will be  $10962'' = 3^\circ 2' 42''$ ; the correction, on account of the excentricity, being only  $21''$ .

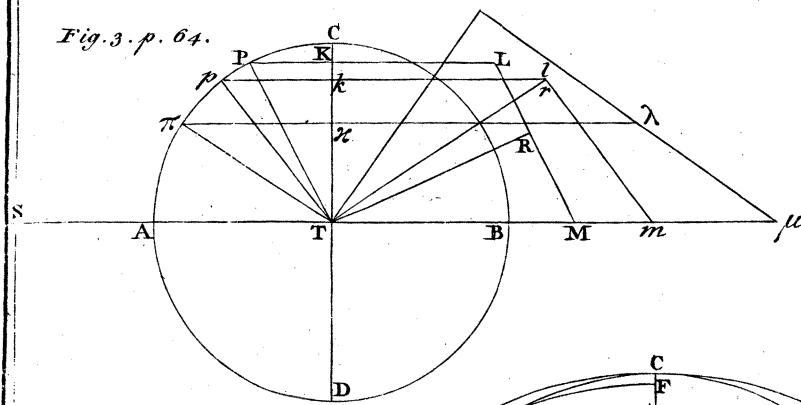
Multiply  $3^\circ 2' 42''$  by 1.080853, and the product  $3^\circ 17' 28''$  is the mean motion of the apogee, in a synodical month; exceeding the quantity marked in the tables by no more than  $4''$ .

17. Of the obliquity of the moon's orbit, to the plane of the ecliptic, we take no notice: because, altho', absolutely speaking, a force in that plane, referred to the moon's orbit, would, thence, be diminish'd by about  $\frac{3}{1000}$  parts; yet, in the present case, the effect of the obliquity is included in the first determination of the quantity  $c$ , from the periodical times of the earth and moon; all but what belongs to the corrections; and which is only  $110''.x003 = 0''.33$ , to be subtracted.

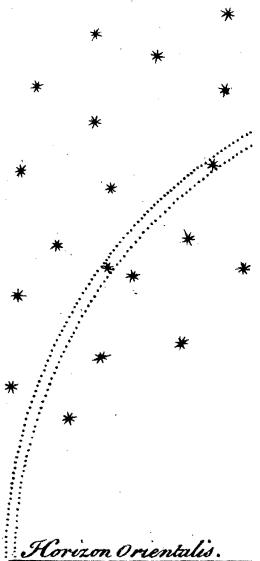
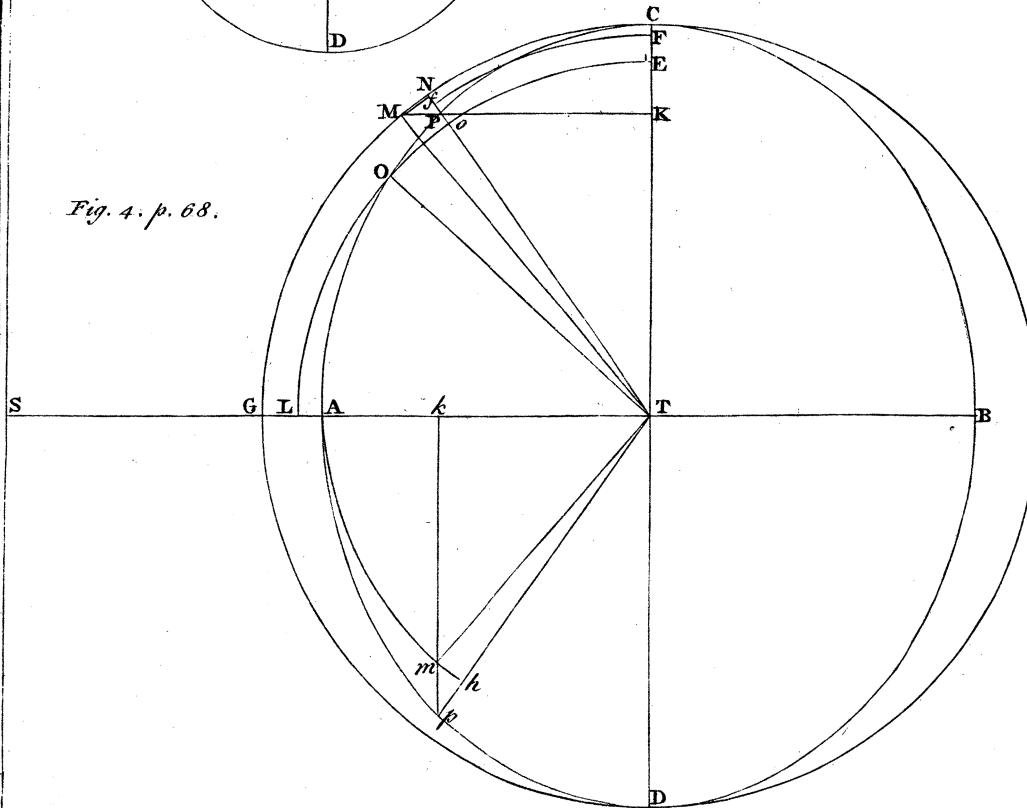
18. The force  $c$  is, itself, the effect of the sun's parallax, and the total effect; excepting only a small difference between his action on the moon, when she is waxing or waning, and when she is in the other half of her orbit; neglected as altogether inconsiderable.

Upon the whole, we may conclude, that, in this, as in the other phænomena of the celestial motions, the principles and rules of Sir Isaac Newton are fully confirmed and verified.

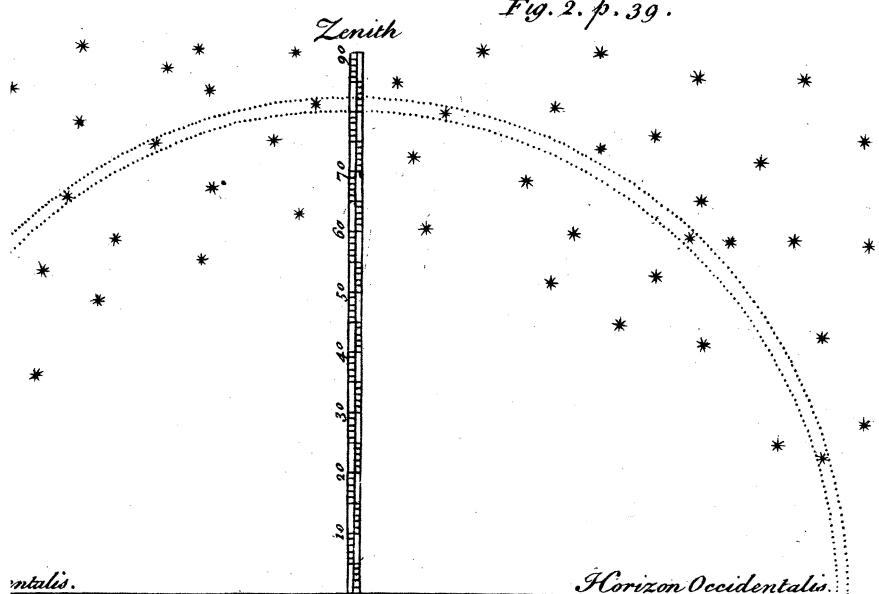
*Fig. 3. p. 64.*



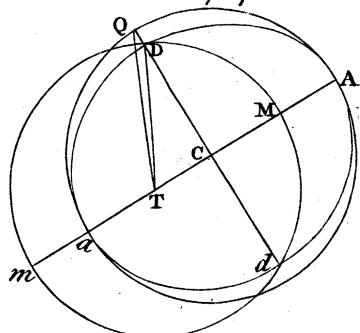
*Fig. 4. p. 68.*



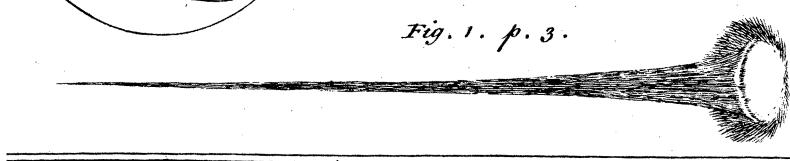
*Fig. 2. p. 39.*



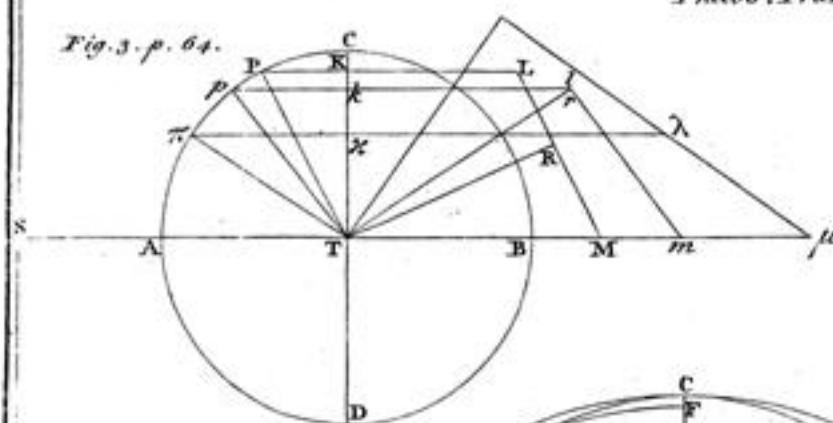
*Fig. 5. p. 73.*



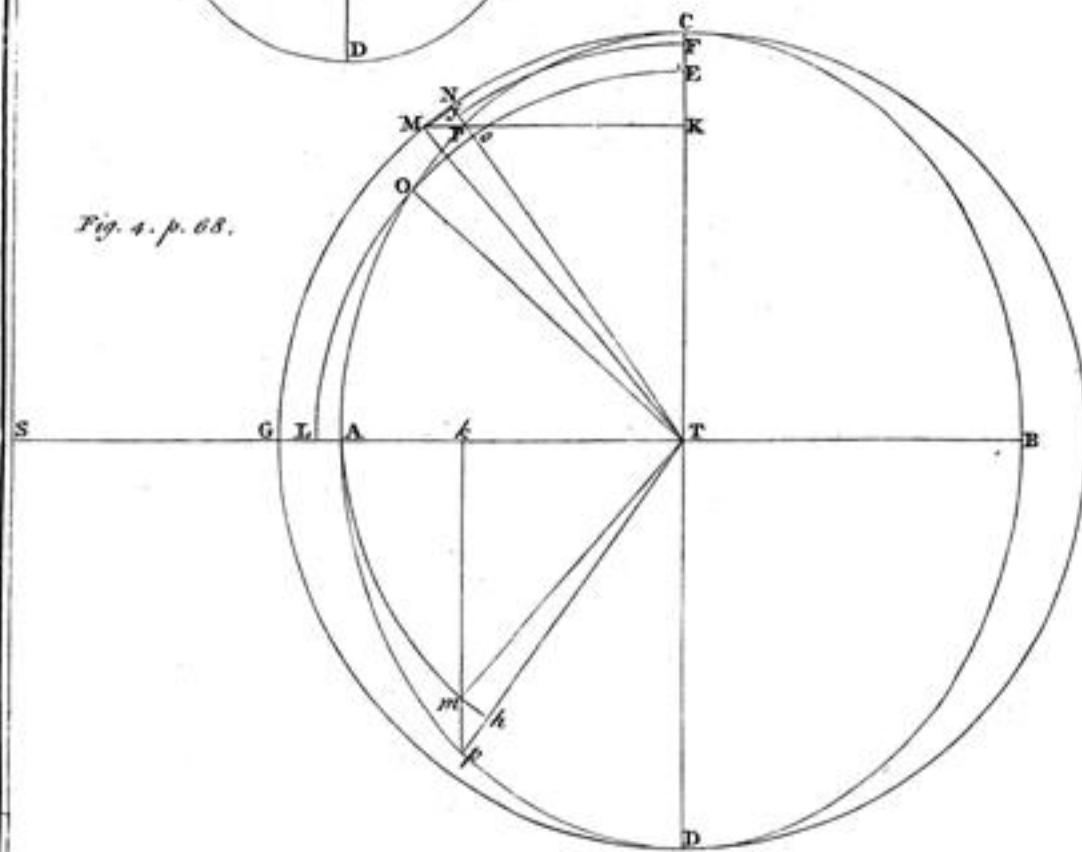
*Fig. 1. p. 3.*



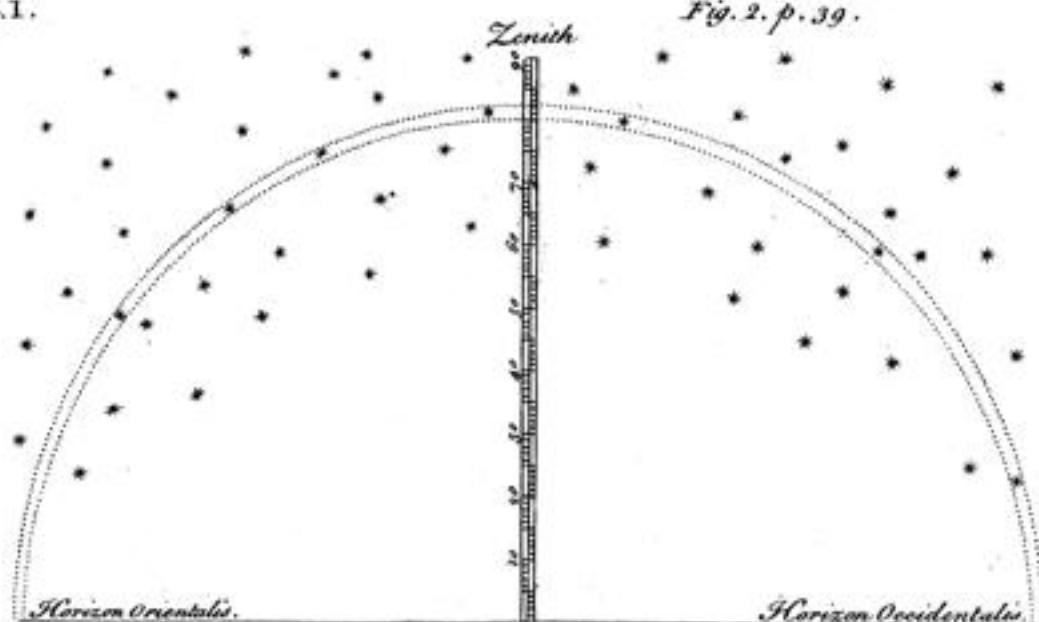
*Fig. 3. p. 64.*



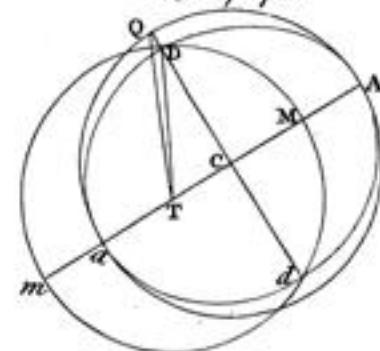
*Fig. 4. p. 68.*



*Fig. 2. p. 39.*



*Fig. 5. p. 73.*



*Fig. 1. p. 3.*

